INVERSE PROBLEMS:

MATHEMATICAL BACKGROUND

or

STATISTICAL METHODS IN

BIOMEDICAL APPLICATIONS

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STATISTICAL INFERENCE

Retrieval of information by indirect observations

$$\begin{aligned} x &= \text{unknown interesting quantity} \\ y &= \text{observable} \end{aligned}$$

Problem: Estimate variable x based on observed y.

Example 1: (Interdependence derived from physics):

$$\begin{aligned} x &= \text{temperature of the air} \\ y &= \text{length of a metal rod} + \text{noise} \end{aligned}$$

Model:

$$y = Kx + \varepsilon.$$

Example 2: (Interdependence derived from previous observations):

$$\begin{aligned} x &= \text{temperature of the air} \\ y &= \text{time of the day} \end{aligned}$$

Scatter plot of previous observations.

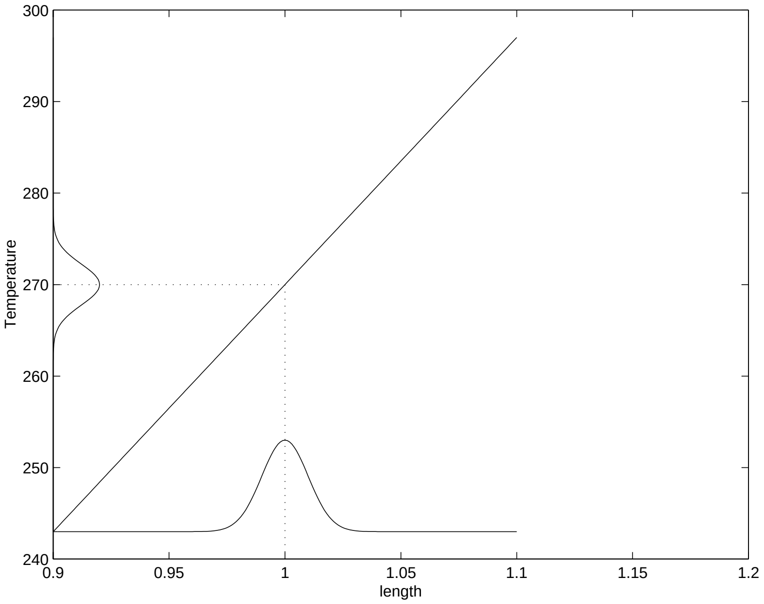

Figure 1

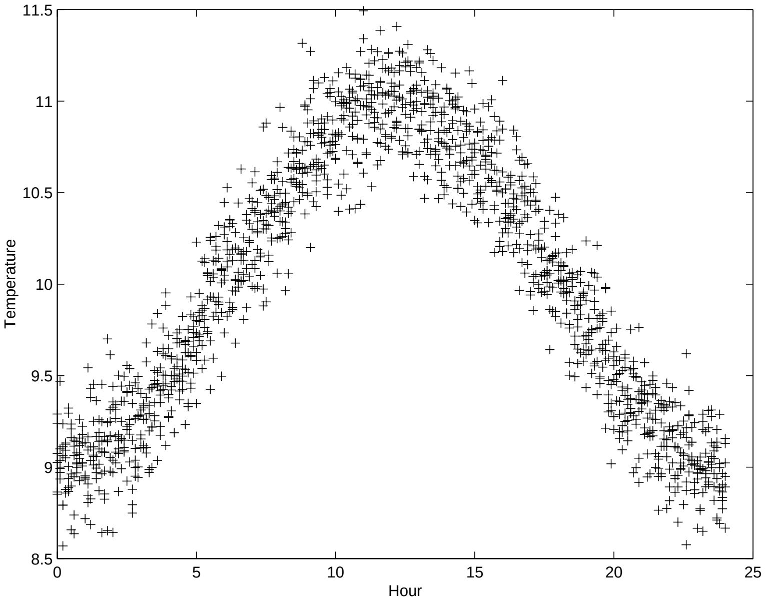

Figure 2

Fundamental consepts: Joint probability distribution, conditional distribution

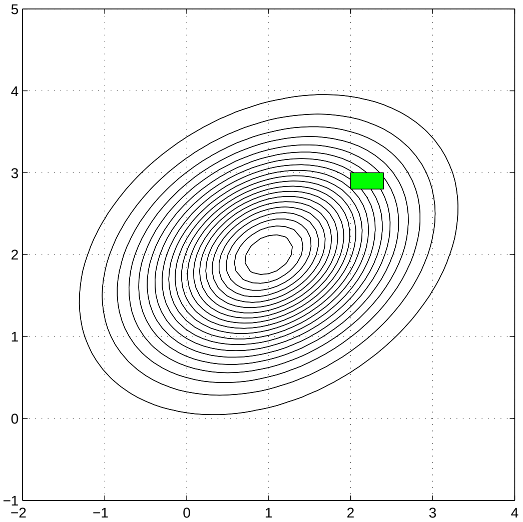

Figure 3: Contour plot of a Gaussian distribution. Probability of the random variable to reside in the shaded rectangle is the integral over the set.

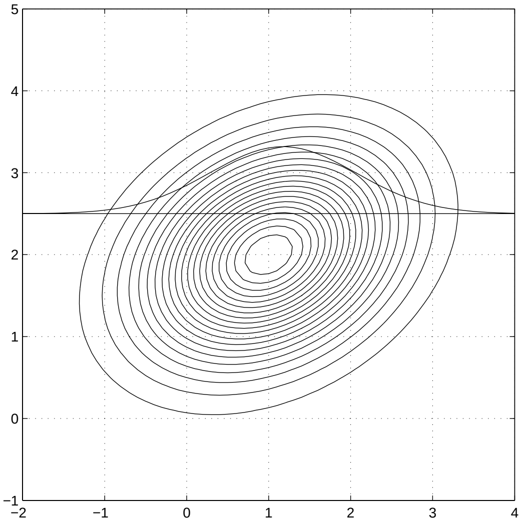

Figure 4: Gaussian distribution. The random variable X_2 is fixed at the value $x_2 = 2.5$. The conditional distribution of the variable X_1 is obtained by cutting the joint probability distribution alon $x_2 = 2.5$.

STATISTICAL INVERSION THEORY

1. Physical quantities are modelled as **random variables**.
2. Random variables define **probability distributions**.
3. The solution of an inverse problem is a **conditional probability distribution** (conditioned on the observation) called **posterior distribution**.
4. Single estimate calculation is based on the posterior distribution.

The most popular estimates are **maximum a posteriori estimator**and **conditional expectation**.

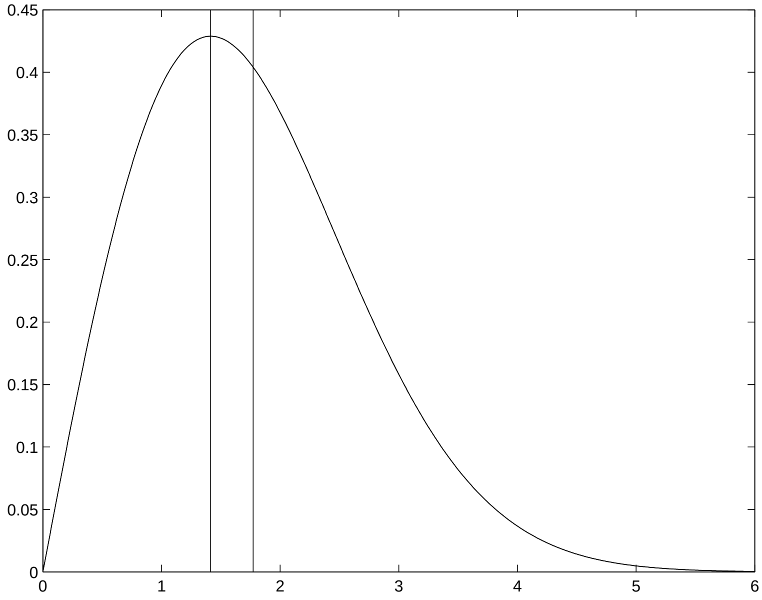

Figure 5: The maximum and the expectation are marked. Notice that due to the skewness of the distribution, they do not coincide.

The statistical model should contain

1. **The physical model** interrelating the quantities (if available).
2. **Information of the statistics** of, noise, modelling errors etc.
3. **Prior information** of the quantities (based on previous experience, physics, common sense etc.)

Example: Electrical Impedance Tomography.

Attach contact electrodes on the surface of a body.

Inject electric currents, measure voltages.

Estimate the resistivity (or impedance) distribution in a body from these estimates.

Unknown: Resistivity $\rho = (\rho_1, \rho_2, \ldots, \rho_N)$. (Pixels)

Data: Voltages $V = (V_1, V_2, \ldots, V_L)$, currents $I = (I_1, I_2, \ldots, I_L)$.

Model (Ohm's law):

$$V = R(\rho)I + n,$$

where $R(\rho)$ is a resistance matrix computed from a differential equation (e.g. Finite Element Method)

$n =$ measurement noise.

Statistics: Noise is a random variable with probability density $\pi_{\text{noise}}(n)$.

Assume for a moment that ρ were known. Then, V is a random variable with probability distribution

$$\pi(V \mid \rho) = \pi_{\text{noise}}(V - R(\rho)I).$$

Prior distribution of ρ is $\pi_{\text{pr}}(\rho)$.

Contains all the information of rho known to us prior to the measurement (e.g. positivity, boundedness, possible smoothness, anatomical information).

If ρ is independent of n, probability theory says that the **joint probability distribution** of ρ and V is

$$\pi(\rho, V) = \pi_{\text{pr}}(\rho)\pi_{\text{noise}}(V - R(\rho)I).$$

Posterior distribution = **Bayes formula**,

$$\begin{aligned}\pi(\rho \mid V = V_{\text{measured}}) &\sim \pi(\rho, V = V_{\text{measured}}) \\ &= \pi_{\text{pr}}(\rho)\pi_{\text{noise}}(V_{\text{measured}} - R(\rho)I) \\ &= \pi_{\text{post}}(\rho).\end{aligned}$$

In statistical inversion theory, *this is the complete solution of the inverse problem.*

ESTIMATORS

Maximum a posteriori estimator (MAP): Find $\rho = \rho_{\mathrm{MAP}}$ such that

$$\pi_{\mathrm{post}}(\rho_{\mathrm{MAP}}) = \max \pi_{\mathrm{post}}(\rho).$$

Conditional expectation: Calculate

$$\overline{\rho} = \int_{\mathbb{R}^N} \rho \; \pi_{\mathrm{post}}(\rho) d^N \rho,$$

i.e., componentwise (pixelwise),

$$\overline{\rho}_j = \int_{\mathbb{R}^N} \rho_j \; \pi_{\mathrm{post}}(\rho) d^N \rho, \quad 1 \leq j \leq N.$$

PROBLEMS

The focus of the current research is on two questions:

1. How to implement prior information?
2. How to compute single estimators?

PRIORS: GENERAL SCHEME

Write

$$\pi_{\mathrm{pr}}(\rho) \sim e^{-A(\rho)},$$

where $A(\rho)$ is a function that is *small* for typical admissible resistivities, and *large* for typical unadmissible resistivities.

Example: MRI-guided EIT: Admissible conductivities jump across organ boundaries, unadmissible conductivities are smooth across the boundaries.

ESTIMATORS: GENERAL SCHEME

Given $\pi_{\text{post}}(\rho)$, generate a large ensemble of samples

$$\{\rho^{(1)}, \rho^{(2)}, \ldots, \rho^{(K)}\}$$

that are distributed according to the probability density $\pi_{\text{post}}(\rho)$. Approximate

$$\begin{aligned}\overline{\rho} &= \int_{\mathbb{R}^N} \rho \, \pi_{\text{post}}(\rho) d^N \rho \\ &\approx \frac{1}{K} \sum_{k=1}^{K} \rho^{(k)}.\end{aligned}$$

Ensemble generation by *Markov Chain Monte Carlo* (MCMC) methods.

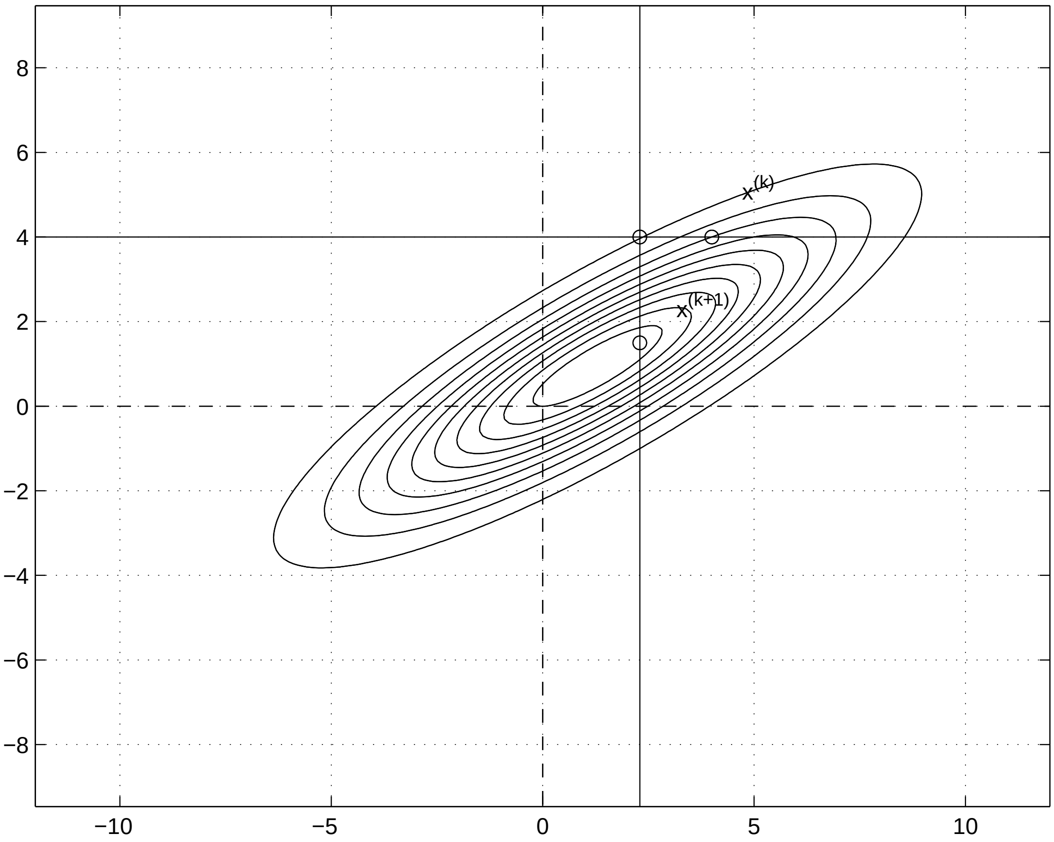

Figure 6: Gaussian distribution with positivity constraints, two consecutive samples.

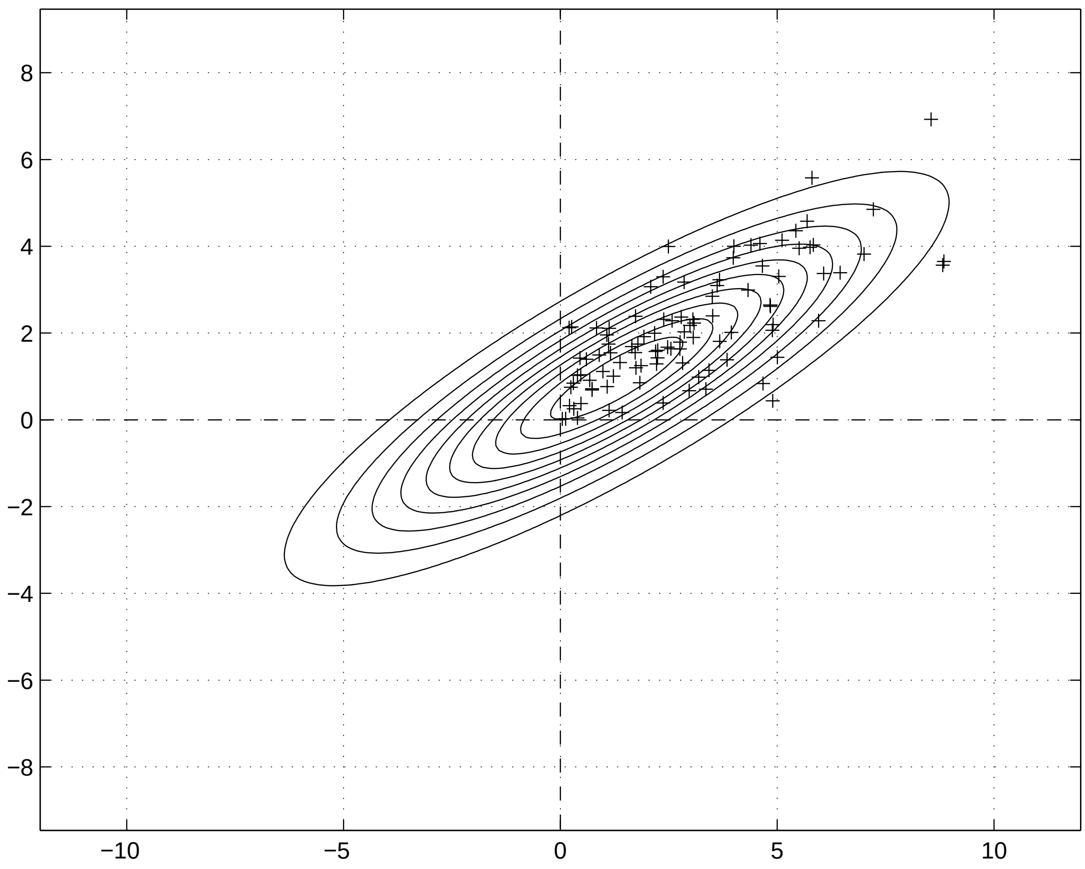

Figure 7: Gaussian distribution with positivity constraints, 100 sample points by Gibbs sampler.

EXAMPLE (continued):

Gaussian additive noise:

$$\pi_{\mathrm{noise}}(n) \sim \exp\left(-\frac{1}{2}n^{\mathrm{T}}C^{-1}n\right),$$

where $C \in \mathbb{R}^{L\times L}$ is the noise covariance matrix.

$$\pi(V \mid \rho) \sim \exp\left(-\frac{1}{2}(V - R(\rho)I)^{\mathrm{T}}C^{-1}(V - R(\rho)I)\right).$$

Selection of the prior: Try

$$\pi_{\mathrm{pr}}(\rho) \sim \exp\left(-\alpha \sum_{n=1}^{N} |\rho_n - \rho_0|\right).$$

It turns out that the function

$$A(\rho) = \sum_{n=1}^{N} |\rho_n - \rho_0|$$

is small, when only few pixels differ from ρ_0.

Suitable e.g. for small tumor detection.

Discussion of results: See

J. Kaipio, V. Kolehmainen, E. Somersalo and M. Vauhkonen: Statistical Inversion Methods in Electrical Impedance Tomography. (To appear in *Inverse Problems*, 2000)